Geometry of fractals

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Fractal and Related Fields

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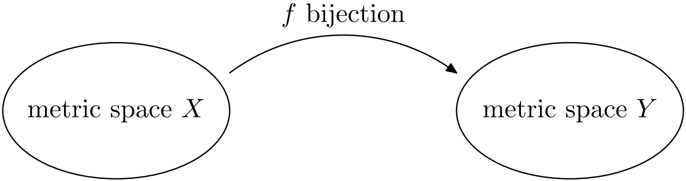
Background 1: Geometry in view of Lipschitz equivalence

Gromov: [Metric Structures for Riemannian and Non-Riemannian Spaces]
"isometry" leads to a poor and rather boring category.
"continuity" takes us out of geometry to the realm of pure topology.
We mediate between the two extremes by emphasizing bi-Lipschitz maps.

Falconer: [On the Lipschitz equivalence of Cantor sets.1992]
Topology may be considered as the study of equivalence classes of sets under homeomorphism, and then fractal geometry is sometimes regarded as the study of equivalence classes of sets under bi-Lipschitz mappings.

Lipschitz equivalence



$$C^{-1} \leq \frac{d_Y\big(f(x_1), f(x_2)\big)}{d_X(x_1, x_2)} \leq C$$

We say that X and Y are Lipschitz equivalent, denoted by $X \simeq Y$.

Question

$$X \simeq Y \Rightarrow dim_H X = dim_H Y,$$

Two self-similar sets with the same dimension need not be bilipschitz equivalent.

Example: [Falconer & Marsh, 1992].

Let $3r^{\log 2/\log 3} = 1$, suppose a self-similar set is generated by similitudes

$$rx, rx + (1-r)/2, rx + 1 - r,$$

then this self-similar set and the Cantor ternary set have the same Hausdorff dimension $\log 2/\log 3$, but they are not Lipschitz equivalent .

Question:

$$dim_H X = dim_H Y \overset{when?}{\Rightarrow} X \overset{q}{\simeq} Y.$$

Here X, Y can be taken as self-similar sets, self-conformal sets, graph-directed sets, Moran sets, etc.

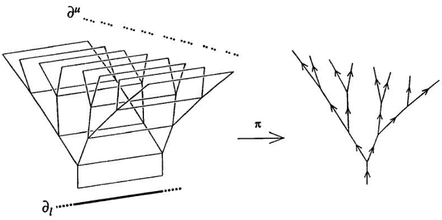
Background 2: Geometric Group Theory

Sometimes, to classify geometric groups (by quasi-isometry), we shall solve the Lipschitz equivalence of their fractal boundaries.

For example, two Baumslag-Solitar groups are quasi-isometric if and only if their boundaries, which are self-similar sets, are Lipschitz equivalent.

The solvable Baumslag-Solitar groups. The solvable Baumslag-Solitar groups $BS(1,n)$ are given by the presentation

$$\mathrm{BS}(1,n) = \langle a, b \mid aba^{-1} = b^n \rangle \ .$$



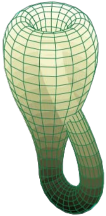
Background 3: Embedding in Topology

The Cantor ternary set can be embedded into any perfect set.

Differential Manifold:

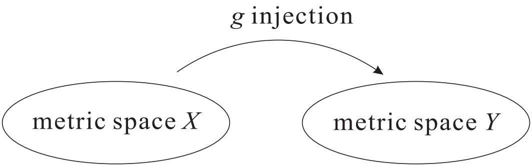
Whintey embedding theorem

Nash embedding theorem



For example, the Klein bottle can not be embedded into $\mathbb{R}^3$

Lipschitz embedding



$$D^{-1} \le \frac{d_Y\big(g(x_1), g(x_2)\big)}{d_X(x_1, x_2)} \le D$$

We say that (X, d_X) can be embedded into (Y, d_Y), denoted by $X \hookrightarrow Y$.

Question

It is easy to see that

$$X \hookrightarrow Y \Rightarrow \; dim_H X \leq dim_H Y.$$

For two fractals in the same fractal class, e.g. two self-similar sets, we can imagine that one fractal with lower dimension can be embedded into the another with higher dimension.

Question:

$$dim_H X \leq dim_H Y \overset{when?}{\Rightarrow} X \hookrightarrow Y.$$

Here X, Y can be taken in the classical fractal classes:

a. self-similar sets
b. self-conformal sets
c. graph-directed sets
d. Moran sets, etc.

Lipschitz equivalence of self-similar sets with SSC: necessary condition

Suppose E, F are self-similar sets satisfying SSC (strong separation condition) with ratio sets $\{r_i\}_{i=1}^n$ and $\{t_j\}_{j=1}^m$ respectively.

Falconer and Marsh (1992) gave a necessary condition for E and F to be Lipschitz equivalent:

If $E \simeq F$, then
(i) $\mathbb{Q}(r_1^s, \cdots, r_n^s) = \mathbb{Q}(t_1^s, \cdots, t_m^s)$;
(ii) There are positive integers p and q such that

$$sgp(r_1^p, \cdots, r_n^p) \subset sgp(t_1, ..., t_m),$$

$$sgp(t_1^q, \cdots, t_m^q) \subset sgp(r_1, \cdots r_m),$$

where $sgp(a_1, \cdots, a_k)$ is the multiplicative semigroup generated by $\{a_1, \cdots, a_k\}$.

Lipschitz equivalence of self-similar sets: notations

Write $\rho_j = t_j^s$ for $j = 1, \cdots m$. Then $\sum_{j=1}^m \rho_j = 1$.
Let $\Sigma_m = \{1, \cdots, m\}^\infty$ be a symbolic system equipped with the Bernoulli measure $\mu = (\rho_1, \cdots, \rho_m)$.
Σ_m^*: the set of all finite words $\cup_{i=1}^\infty \{1, \cdots, m\}^i$.
The cylinder $[i_1 \cdots i_l] = \{j_1 \cdots j_l \cdots \in \Sigma_m : j_1 \cdots j_l = i_1 \cdots i_l\}$. Then

$$\mu([i_1 \cdots i_l]) = \Pi_{t=1}^l \rho_{i_t}.$$

Let $\Lambda_{\mathcal{T}} = \{\Omega = \cup_{u=1}^k [\mathbf{i}_u^*] : k \in \mathbb{N}, \mathbf{i}_u^* \in \Sigma_m^*$ for all u and $[\mathbf{i}_u^*] \cap [\mathbf{i}_v^*] = \varnothing$ for any $u \neq v\}$. That means any element of $\Lambda_{\mathcal{T}}$ is the union of finite cylinders.
Given a word $i_1 \cdots i_k$ and a subset A of Σ_m, let

$$i_1 \cdots i_k A = \{i_1 \cdots i_k j_1 \cdots j_l \cdots : j_1 \cdots j_l \cdots \in A\}.$$

For $\Omega, \Omega' \in \Lambda_{\mathcal{T}}$, we denote $\Omega \prec \Omega'$, if either $\Omega = \Omega'$ or there is a word $i_1 \cdots i_k$ such that $\Omega = i_1 \cdots i_k \Omega'$.
Fix n, let $\Gamma_k = \{(i, j) : 1 \leq i \leq k, 1 \leq j \leq n\}$ for $k \geq 1$.

Lipschitz equivalence of self-similar sets with SSC: necessary and sufficient condition

Xi, 2010, Math Z.

Theorem: Suppose $\mathcal{R} = \{r_i\}_{i=1}^n$ and $\mathcal{T} = \{t_j\}_{j=1}^m$ are ratios sets with $\sum r_i^s = \sum t_j^s = 1$. Let E, F be self-similar sets satisfying SSC with ratio sets $\{r_i\}_{i=1}^n$ and $\{t_j\}_{j=1}^m$ respectively. Then $E \simeq F$ if and only if there exist $\Omega_1, \Omega_2, \cdots, \Omega_k \in \Lambda_{\mathcal{T}}$ for some integer k, $\{\Omega_{i,j}\}_{(i,j)\in\Gamma_k} \subset \Lambda_{\mathcal{T}}$ and $\gamma : \Gamma_k \to \{1, \cdots, k\}$ such that
(1) $\Omega_{i,j} \prec \Omega_{\gamma(i,j)}$ for every $(i,j) \in \Gamma_k$;
(2) For every $1 \le i \le k$, $\Omega_i = \cup_{j=1}^n \Omega_{i,j}$, which is a disjoint union;
(3) For every $(i,j) \in \Gamma_k$,

$$\mu(\Omega_{i,j})/\mu(\Omega_i) = r_j^s.$$

Remark: By this theorem, given a self-similar set E with SSC, there is an algorithm to obtain all self-similar sets which are Lipschitz equivalent to E.

Lipschitz equivalence of self-similar arcs

Wen et al, 2003, Israel J. Math.

Definition: An arc means a homeomorphic image of [0,1].
An arc γ is called a self-similar arc, if γ is generated by a contractive similitudes $\{S_i\}_{i=1}^m$ satisfying
(1) $S_i(\gamma) \cap S_j(\gamma)$ is a singleton for $|i-j| = 1$;
(2) $S_i(\gamma) \cap S_j(\gamma) = \varnothing$ for $|i-j| > 1$.

For example, the Koch curve is a self-similar arc.

Theorem: There are two self-similar arcs γ_1, γ_2 in the plane such that $dim_H \gamma_1 = dim_H \gamma_2$, but γ_1 and γ_2 are not Lipschitz equivalent.

Lipschitz equivalence of {1,3,5}-{1,4,5} sets

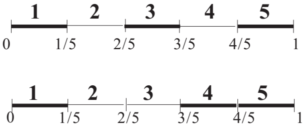
David and Semmes: Fractured Fractals and Broken Dreams, 1997

David and Semmes proposed a special question, the so called $\{1,3,5\}$-$\{1,4,5\}$ problem.
$\{1,3,5\}$-$\{1,4,5\}$ sets:

$$E_{1,3,5} = (E_{1,3,5}/5) \cup (E_{1,3,5}/5 + 2/5) \cup (E_{1,3,5}/5 + 4/5),$$
$$E_{1,4,5} = (E_{1,4,5}/5) \cup (E_{1,4,5}/5 + 3/5) \cup (E_{1,4,5}/5 + 4/5).$$

David and Semmes asked:

whether $E_{1,3,5}$ and $E_{1,4,5}$ are Lipschitz equivalent or not.



Initial pattern for {1,3,5} set

Initial pattern for {1,4,5} set

Lipschitz equivalence of generalized {1,3,5}-{1,4,5} sets

Rao, Ruan, Xi, 2006, Comptes Rendus Mathematics

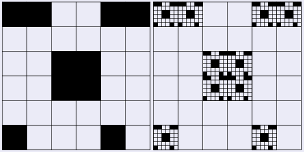
Theorem: $E_{1,3,5} \simeq E_{1,4,5}$.

Xi, Xiong, 2010, Comptes Rendus Mathematics

Theorem: Let $A, B \subset \{0, \dots, m-1\}^d$ and

$$E_A = \bigcup_{a \in A} m^{-1}(E_A + a), \quad E_B = \bigcup_{b \in B} m^{-1}(E_B + b)$$

be two totally disconnected self-similar sets, then $E_A \simeq E_B$ if and only if card $A =$ card B.



Lipschitz equivalence of generalized {1,3,5}-{1,4,5} sets

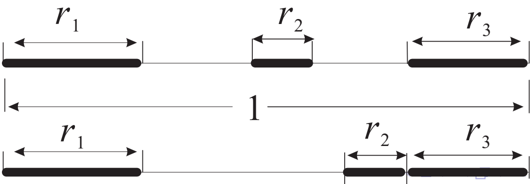
Xi, Ruan, 2007, Science in China

We consider a generalized situation for three different ratios r_1, r_2, r_3. Suppose $S_1(x) = T_1(x) = r_1 x$, $S_3(x) = T_3(x) = (1 - r_3) + r_3 x$, and

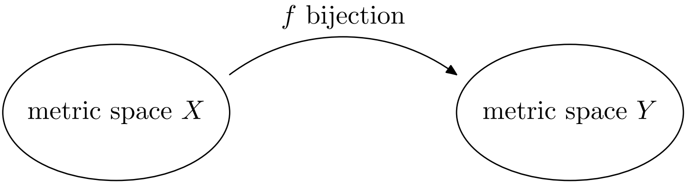
$$S_2(x) = \frac{1 + r_1 - r_2 - r_3}{2} + r_2 x, \;\; T_2(x) = 1 - r_2 - r_3 + r_2 x.$$

Let $F_{\{r_1,r_2,r_3\}}$ and $G_{\{r_1,r_2,r_3\}}$ be self-similar sets generated by $\{S_i\}_{i=1}^3$ and $\{T_i\}_{i=1}^3$ respectively.

Theorem: $F_{\{r_1,r_2,r_3\}}$ and $G_{\{r_1,r_2,r_3\}}$ are Lipschitz equivalent if and only $\log r_1 / \log r_3$ is rational.



Quasi-Lipschitz equivalence



$$\frac{\log d_Y(f(x_1), f(x_2))}{\log d_X(x_1, x_2)} \to 1 \text{ uniformly as } d_X(x_1, x_2) \to 0$$

We say that X and Y are quasi-Lipschitz equivalent, denoted by $X \overset{q}{\simeq} Y$.

$$X \overset{q}{\simeq} Y \Rightarrow \; dim_H X = dim_H Y,$$

Question :

$$dim_H X \leq dim_H Y \overset{when?}{\Rightarrow} X \hookrightarrow Y.$$

Result related to quasi-Lipschitz equivalence

L. F. Xi 2007, Israel J. Math.

Suppose E, F are C^1 self-conformal sets satisfying the strong separation condition (SSC). If $\dim_H E = \dim_H F = s$ and $0 < \mathcal{H}^s(E), \mathcal{H}^s(F) < +\infty$, then E and F are quasi-Lipschitz equivalent.

Q. Wang, L. F. Xi 2011, Nonlinearity

Given a metric space X, we say a subset E of X is uniformly disconnected if for any $x \in E, R > 0$, there is a subset $B \subset E$ such that $dist(B, E \setminus B) > R/c$ and $E \cap B(x, R/c) \subset B \subset B(x, R)$, where $c \geq 1$ is a constant independent of any x, R.

Theorem: The compact and uniformly disconnected sets E and F are s-regular and t-regular, respectively. Then $E \overset{q}{\simeq} F$ if and only if $s = t$.

Embedding: Result of Mattila and Saaranen

Given a metric space X, we say a subset F of X is Ahlfors-David s-regular, if there exists a constant $C > 0$ and a Borel measure μ supported on F, such that

$$C^{-1}r^s \leq \mu(B(x,r) \cap F) \leq Cr^s$$

for all $x \in F, 0 < r \leq |F|$ and $r < \infty$. We have $dim_H X = s$.

Mattila and Saaranen, 2009, Ann. Acad. Sci. Fenn. Math.

Theorem: For any bounded s-regular set E and $t \in (0,s)$, there is a t-regular subset F of E and a bilipschitz map $f : F \to C_{t,E}$, where $C_{t,E} \subset \mathbb{R}^n$ is a t-dimensional self-similar set generated by 2^n similitudes with the same ratio.

BPI equivalence

David and Semmes: Fractured Fractals and Broken Dreams, 1997

David and Semmes introduced the BPI (*big pieces of itself*) equivalence, which is weaker than Lipschitz equivalence.

In particular for self-similar sets, they proved that:
Theorem: The self-similar sets F and F' with OSC are BPI equivalent, if and only if there are $K \subset F$, $K' \subset F'$ such that K and K' are Lipschitz equivalent with $\mathcal{H}^s(K), \mathcal{H}^s(K') > 0$.

Question: Under what condition for self-similar (self-conformal) sets,

$$BPI \quad euqivalence \Leftrightarrow Lipschitz \ equivalence?$$

Embedding: Result of Llorente and Mattila

Llorente and Mattila, 2010, Nonlinearity

They consider the self-conformal sets, which are much more general than self-similar sets in Euclidean spaces.

Theorem: For conformal systems $\{f_1, ..., f_N\}$ and $\{g_1, g_2\}$ satisfying SSC with invariant sets $E = \cup_{i=1}^N f_i(E)$ and $F = \cup_{j=1}^2 g_j(F)$,

$$BPI \;\; euqivalence \Leftrightarrow Lipschitz \;\; equivalence.$$

In the added notes, they pointed out the conclusion is still valid if $F = \cup_{j=1}^M g_j(F)$ is a *self-similar* set in $\mathbb{R}^1$ with SSC, where the integer $M \geq 2$ is arbitrary.

Embedding: in the case of the different dimensions

Deng, Wen, Xi and Xiong, Journal d'Analyse Mathematique, in press

Theorem 1: Suppose E_1 and E_2 are self-similar sets in metric spaces with

$$\dim_H E_1 < \dim_H E_2.$$

If E_1 satisfies the strong separation condition, then $E_1 \hookrightarrow E_2$.

Remark: In the above theorem, we don't need the strong separation for E_2.

Remark: We consider the self-similar sets in complete metric spaces, not only in Euclidean spaces.

Embedding: in the case of the same dimension

Deng, Wen, Xi and Xiong, Journal d'Analyse Mathematique, in press

Theorem 2: Suppose F and F' are self-similar sets in complete metric spaces satisfying the strong separation condition, and

$$\dim_H F = \dim_H F' = s.$$

Then $F \hookrightarrow F'$ if and only if F and F' are Lipschitz equivalent.

Embedding: BPI and Lipschitz equivalence

Deng, Wen, Xi and Xiong, Journal d'Analyse Mathematique, in press

Theorem 3: Keep the same assumption as in Theorem 2. If there are $K \subset F$ and $K' \subset F'$ with $\mathcal{H}^s(K), \mathcal{H}^s(K') > 0$, such that K and K' are Lipschitz equivalent, then F and F' are Lipschitz equivalent.

That means for self-similar set with SSC,

$$BPI \quad euqivalence \Leftrightarrow Lipschitz \ equivalence.$$

Embedding: s-structure

Deng, Wen, Xi and Xiong, Journal d'Analyse Mathematique, in press

Let C, δ, s be positive numbers. A sequence $\{\Phi_k\}_{k\geq 0}$ is called *controlled by* (C, δ, s) provided

a. For any $k \geq 1$, Φ_k is a collection of words with length k, by convention, $\Phi_0 = \{\emptyset\}$, where $\emptyset$ is the empty word;
b. If $i_1 \cdots i_{k-1} i_k \in \Phi_k$, then $i_1 \cdots i_{k-1} \in \Phi_{k-1}$;
c. For any $k_2 > k_1 \geq 0$ and any $i_1 i_2 \cdots i_{k_1} \in \Phi_{k_1}$,

$$card\{i_1 i_2 \cdots i_{k_1} \cdots i_{k_2} \in \Phi_{k_2}\} \leq C(\delta^{-s})^{k_2 - k_1}.$$

Embedding: s-structure

Deng, Wen, Xi and Xiong, Journal d'Analyse Mathematique, in press

Let (X, d) be a metric space, $C > 1, 0 < \delta < 1, s > 0$, and let $\{\Phi_k\}_{k\geq 1}$ be a sequence controlled by (C, δ, s). Suppose $E \subset X$ and for any $k \in \mathbb{N}$, there is a decomposition of E with respect to the sequence $\{\Phi_k\}_{k\geq 0}$:

$$E = \bigcup_{i_1 i_2 \cdots i_k \in \Phi_k} E^{i_1 i_2 \cdots i_k}.$$

We say that the set E has an *s-structure* if for any $i_1 i_2 \cdots i_k \in \Phi_k$, we have

1) $E^{i_1 i_2 \cdots i_k} = \bigcup_{i_1 i_2 \cdots i_k j \in \Phi_{k+1}} E^{i_1 i_2 \cdots i_k j}$;
2) $|E^{i_1 i_2 \cdots i_k}| \leq C\delta^k$;
3) $d(E^{i_1 i_2 \cdots i_k}, E^{j_1 j_2 \cdots j_k}) \geq C^{-1}\delta^k$ whenever $i_1 \cdots i_k \neq j_1 \cdots j_k$.

Embedding: s-structure

Deng, Wen, Xi and Xiong, Journal d'Analyse Mathematique, in press

Proposition 1: For $s > 0$, the following sets have s-structures:
(1) Bounded s-regular sets with $s \in (0, 1)$;
(2) $C^{1+\alpha}(\alpha > 0)$ self-conformal sets with dimension s;
(3) Self-similar sets in metric spaces satisfying SSC with dimension s;
(4) Graph-directed sets (on a transitive graph) satisfying SSC with dimension s;
(5) Homogeneous Moran sets in $\mathcal{M}(J, \bar{n}, \bar{c})$ $(\bar{n}\bar{c} < 1)$ with dimension s.

Proposition 2: If $E \subset X$ has an s-structure and $F \subset Y$ is t-regular with $s < t$, then $E \hookrightarrow F$.

Self-similar sets in complete metric space

Deng, Wen, Xi and Xiong, Journal d'Analyse Mathematique, in press

Proposition 3: If F is a self-similar set in a complete metric space with $\dim_H F = t$. For any $\varepsilon > 0$, there is a self-similar set F_ε satisfying the strong separation condition (SSC) such that

$$F_\varepsilon \subset F \text{ and } \dim_H F_\varepsilon \in (t - \varepsilon, t].$$

Hence F_ε is t_ε-regular with $t_\varepsilon \in (t - \varepsilon, t]$.

Proposition 4: If F is a self-similar set with SSC in a complete metric space, then there is a self-similar set with SSC in Euclidean space such that E and F are Lipschitz equivalent.

Remark Theorem 1 follows from Propositions 1(3), 2, 3.
Remark Theorem 3 follows from the technique in [Xi, 2010, Math Z]. Thus Theorem 2 follows.

Questions

How about the embedding in the following cases:
(1) E and F are self-similar sets with the open set condition;
(2) E and F are Moran sets which not in the category of IFS;
(3) E and F are Ahlfors-David regular sets.

Theorem 3 shows that self-similar sets satisfying SSC are BPI equivalent if and only if they are Lipschitz equivalent.

Open Problem: Under what condition, BPI equivalence implies Lipschitz equivalence for self-similar sets.