Gaussian Random Fields: Spectral Measures and Fine Properties

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Fractals and Related Fields, II

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Outline

- Gaussian fields with stationary increments
- Analytic properties of Gaussian fields
- Fractal properties of Gaussian fields
- Spectral condition for strong local nondeterminism

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an $\mathbb{R}^d$-valued random field defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big). \tag{1}$$

It is of interest to study the geometric properties of the following random sets:

- Range $X\left([0,1]^N\right) = \left\{X(t) : t \in [0,1]^N\right\}$.
- Graph $\mathrm{Gr}X\left([0,1]^N\right) = \left\{(t, X(t)) : t \in [0,1]^N\right\}$.
- Level set $X^{-1}(x) = \left\{t \in \mathbb{R}^N : X(t) = x\right\}$.
- Excursion set $X^{-1}(F) = \left\{t \in \mathbb{R}^N : X(t) \in F\right\}, \forall F \subset \mathbb{R}^d$.
- $\cdots$

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They arise as

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1. Gaussian fields with stationary increments

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and $X(0) = 0$. If $R(s,t) = \mathbb{E}\big[X(s)X(t)\big]$ is continuous, then $R(s,t)$ can be written as

$$R(s,t) = \int_{\mathbb{R}^N} \big(e^{i\langle s,\lambda\rangle} - 1\big)\big(e^{-i\langle t,\lambda\rangle} - 1\big)\,\Delta(d\lambda),$$

where $\Delta(d\lambda)$ is a Borel measure which satisfies

$$\int_{\mathbb{R}^N} \frac{\|\lambda\|^2}{1+\|\lambda\|^2}\,\Delta(d\lambda) < \infty. \qquad (2)$$

The measure Δ is called the *spectral measure* of X.

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It follows that X has the stochastic integral representation:

$$X(t) \stackrel{d}{=} \int_{\mathbb{R}^N} \left(e^{i\langle t,\lambda\rangle} - 1\right) \mathcal{M}(d\lambda),$$

where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions, $\mathcal{M}(d\lambda)$ is a centered complex-valued Gaussian random measure with Δ as its control measure.

For any $h \in \mathbb{R}^N$ we have

$$\mathbb{E}\big(X(t+h) - X(t)\big)^2 = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle h, \lambda\rangle\big)\, \Delta(d\lambda).$$

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- absolutely continuous with density $f(\lambda)$, or
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Some examples

Example 1.1. For $H \in (0,1)$, fractional Brownian motion $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ with index H is a centered Gaussian random field with

$$\mathbb{E}\big(B^H(s)B^H(t)\big) = \frac{1}{2}\Big(\|s\|^{2H} + \|t\|^{2H} - \|s-t\|^{2H}\Big).$$

B^H is H-self-similar, isotropic and has stationary increments.

Its spectral measure has a density function

$$f_H(\lambda) = c(H,N)\|\lambda\|^{-(2H+N)},$$

where $c(H,N) > 0$ is a constant.

Example 1.2. A class of Gaussian fields can be obtained by letting spectral density functions satisfy (2) and

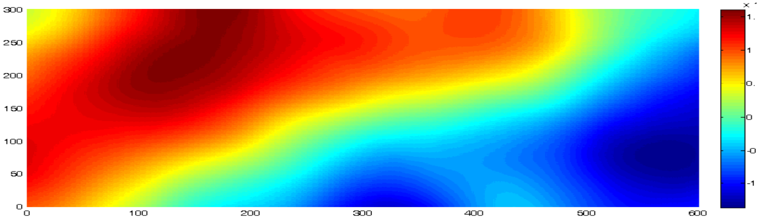
$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^{N} |\lambda_j|^{\beta_j}\right)^{\gamma}}, \qquad \forall\, \lambda \in \mathbb{R}^N,\ \|\lambda\| \geq 1, \tag{3}$$

where $(\beta_1, \ldots, \beta_N) \in (0, \infty)^N$ and

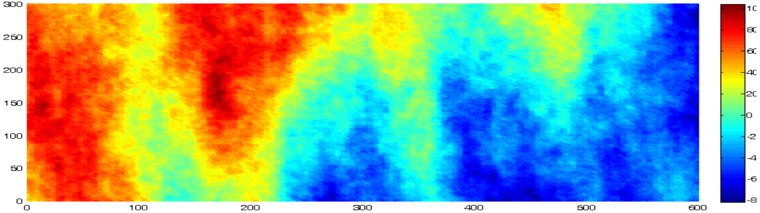
$$\gamma > \sum_{j=1}^{N} \frac{1}{\beta_j}.$$

Gaussian fields with such spectral densities arise in studies of stochastic fractional heat equations and in statistics.

A smooth Gaussian field



A rough Gaussian field



2. Analytic properties of Gaussian fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with spectral density (3). Then

- X is anisotropic and has rich geometric structures.
- $X(t)$ may be smooth in some (or all) directions, "rough" along some other (or all) directions (so that it generates random fractals).

Differentiability

Theorem 2.1 (Xue and X. 2009)

(i) If

$$\beta_j\Big(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\Big) > 2, \tag{4}$$

then the partial derivative $\partial X(t)/\partial t_j$ is continuous almost surely. In particular, if (4) holds for all $1 \le j \le N$, then almost surely $X(t)$ is continuously differentiable.

(ii) If

$$\max_{1\le j\le N} \beta_j\Big(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\Big) \le 2, \tag{5}$$

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Uniform modulus of continuity

Theorem 2.2 (Meerschaert, Wang and X. 2010)

Under condition (5), we have

$$\limsup_{\varepsilon \to 0} \sup_{\substack{s,t \in [0,1]^N, \\ \rho(s,t) \le \varepsilon}} \frac{|X(t) - X(s)|}{\rho(s,t)\sqrt{\log(1 + \rho(s,t)^{-1})}} = c_{2,1}, \qquad (6)$$

where $0 < c_{2,1} < \infty$ is a constant, $\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}$, and for every $1 \le j \le N$,

$$H_j = \frac{\beta_j}{2}\left(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\right) \in (0,1]. \qquad (7)$$

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Ingredients for the proof

- The proof of the upper bound of (6) relies on estimate of the tail probability: For $\varepsilon > 0$ and all $x \geq c_{2,2}\varepsilon\sqrt{\log(1+\varepsilon^{-1})}$,

$$\mathbb{P}\left\{ \sup_{\substack{s,t \in [0,1]^N, \\ \rho(s,t) \leq \varepsilon}} |X(t) - X(s)| \geq x \right\} \leq \exp\left(-c_{2,3}\frac{x^2}{\varepsilon^2}\right).$$

- The proof of the lower bound of (6), and most of the results in this talk rely on the following property of strong local nondeterminism (SLND).

Strong local nondeterminism

Under condition (5),

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^{N} |\lambda_j|^{H_j}\right)^{Q+2}}, \quad \forall \|\lambda\| \geq 1, \tag{8}$$

where $Q = \sum_{i=1}^{N} \frac{1}{H_i}$.

Lemma 2.3 (X. 2009)

$\exists$ constant $c_{2,4} > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in [0, 1]^N$,

$$\operatorname{Var}\Big(X(u) \,\Big|\, X(t^1), \ldots, X(t^n)\Big) \geq c_{2,4} \min_{0 \leq k \leq n} \rho(u, t^k)^2, \tag{9}$$

where $t^0 = 0$.

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Law of the iterated logarithm

Theorem 2.4 (Meerschaert, Wang and X., 2010)

Assume condition (5) holds, then for every $t_0 \in \mathbb{R}^N$

$$\limsup_{r \to 0} \frac{\max_{\rho(0,h) \le r} |X(t_0 + h) - X(t_0)|}{r(\log \log 1/r)^{1/2}} = c_{2,5}, \qquad a.s.,$$

where $0 < c_{2,5} < \infty$ is a constant.

This describes the largest local oscillation of $X(t)$, which is useful for estimating the packing measure of the range and graph.

Chung's law of the iterated logarithm

Theorem 2.5 (Luan and X., 2010)

Assume condition (5) holds, then for every $t_0 \in \mathbb{R}^N$

$$\liminf_{r \to 0} \frac{\max_{\rho(0,h) \le r} |X(t_0 + h) - X(t_0)|}{r(\log\log 1/r)^{-1/Q}} = c_{2,6}, \qquad a.s.,$$

where $0 < c_{2,6} < \infty$ is a constant.

This describes the smallest local oscillation of $X(t)$, which will be useful for estimating the Hausdorff measure of the range and graph.

3. Fractal properties of Gaussian fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d) Gaussian random field defined by (1). That is,

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big).$$

We assume that $X_1, \ldots, X_d$ are independent copies of a real-valued Gaussian field X_0.

We consider first the fractal properties of the range $X([0,1]^N)$, or more generally, $X(E)$, where $E \subseteq [0,1]^N$ is a Borel set.

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3.1 Exact Hausdorff measure of the ranges

Let $\sigma^2(s,t) = \mathbb{E}\left(X_0(s) - X_0(t)\right)^2$ and ρ be the metric on $\mathbb{R}^N$ defined by $\rho(s,t) = \sum_{j=1}^N |s_j - t_j|^{H_j}$ for $s, t \in \mathbb{R}^N$.

Theorem 3.1 (Ayache and X. 2005; X. 2009)

If

$$c_{3,1}\,\rho^2(s,t) \le \sigma^2(s,t) \le c_{3,2}\,\rho^2(s,t), \quad \forall s,t \in [0,1]^N$$

for some constants $c_{3,1} > 0$ and $c_{3,2} < \infty$. Then

$$\dim_{\rm H} X\left([0,1]^N\right) = \min\left\{d;\ \sum_{j=1}^N \frac{1}{H_j}\right\}, \quad \text{a.s.}$$

Question: Is there a function φ such that

$$0 < \mathcal{H}_\varphi\big(X([0,1]^N)\big) < \infty \quad \text{a.s.?}$$

Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be a fractional Brownian motion in $\mathbb{R}^d$ with Hurst index $H \in (0,1)$. Talagrand (1995) showed that, if $N < Hd$, then

$$0 < \mathcal{H}_\varphi\big(B^H([0,1]^N)\big) < \infty \quad \text{a.s.},$$

where

$$\varphi(r) = r^{\frac{N}{H}} \log\log(1/r).$$

Theorem 3.2 (Luan and X., 2010)

Assume that X_0 has stationary increments with spectral density $f(\lambda)$ which satisfies (8).

(i). If $\sum_{j=1}^{N} \frac{1}{H_j} > d$, then a.s. $X([0,1]^N)$ has positive Lebesgue measure (interior points).

(ii). If $\sum_{j=1}^{N} \frac{1}{H_j} < d$, then $\exists$ constants $c_{3,3} > 0$ and $c_{3,4} < \infty$

$$c_{3,3} \le \mathcal{H}_\varphi\left(X\left([0,1]^N\right)\right) \le c_{3,4} \quad \text{a.s.},$$

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Proof of $\mathcal{H}_\varphi\left(X\left([0,1]^N\right)\right) \ge c$

Let μ be the occupation measure of X, defined by

$$\mu(B) = \lambda_N\big\{t \in [0,1]^N : X(t) \in B\big\}, \qquad \forall\, B \in \mathcal{B}(\mathbb{R}^d).$$

Lemma 3.3

For every $t_0 \in [0,1]^N$, let $T_{t_0}(r) = \mu\big(B(X(t_0), r)\big)$. Then

$$\limsup_{r\to 0} \frac{T_{t_0}(r)}{r^{\sum_{j=1}^N \frac{1}{H_j}} \log\log 1/r} \le c < \infty, \qquad \text{a.s.}$$

Sketch of Proof. Since

$$T_{t_0}(r) = \int_{[0,1]^N} \mathbf{1}_{\{\|X(t)-X(t_0)\|\le r\}}\, dt,$$

we use Fubini's theorem to get

$$\begin{aligned}
\mathbb{E}\big(T(r)\big) &= \int_{[0,1]^N} \mathbb{P}\left\{\|X(t)-X(t_0)\| < r\right\} dt \\
&\le \int_{[0,1]^N} \min\left\{1, \left(\frac{r}{\rho(t,t_0)}\right)^d\right\} dt \\
&\le c\, r^Q,
\end{aligned}$$

where $Q = \sum_{j=1}^N H_j^{-1}$.

For all $n \geq 2$,

$$\mathbb{E}\left(T_{t_0}(r)^n\right)$$
$$= \int_{[0,1]^{Nn}} \mathbb{P}\left\{\|X(t^j) - X(t_0)\| \leq r,\ 1 \leq j \leq n\right\} dt^1 \cdots dt^n$$

Using the property of SLND, conditioning and induction, we have

$$\mathbb{E}\left(T_{t_0}(r)^n\right) \leq c^n\, n!\, r^{Qn}.$$

This leads to Lemma 3.3.

Proof of $\mathcal{H}_\varphi\big(X([0,1]^N)\big) \le c$

The covering method was invented by Talagrand (1995), which is different from that for Markov processes.
In our case, we show

- For most of points $t_0 \in [0,1]^N$, there is a sequence $r_n \downarrow 0$, such that $X(B_\rho(t_0, r_n))$ can be covered by a ball of radius $cr(\log\log 1/r)^{-1/Q}$. These are good points.
- For bad points $t_0 \in [0,1]^N$, we can cover $X(B_\rho(t_0, r))$ by a ball of radius $c\, r\sqrt{\log 1/r}$.

The property of SLND plays an important role in the proof.

Question: What about the case $\sum_{j=1}^N \frac{1}{H_j} = d$?

3.2 Hausdorff dimension of $X(E)$

To determine $\dim_{\rm H} X(E)$ for an arbitrary Borel set $E \subset \mathbb{R}^N$, it is not enough to use $\dim_{\rm H} E$.

One needs to use a Hausdorff-type dimension, $\dim_{\rm H}^{\rho} E$, on the metric space $(\mathbb{R}^N, \rho)$.

Theorem 3.4 (Wu and X., 2007; X., 2009)

For any Borel set $E \subseteq \mathbb{R}^N$, we have

$$\dim_{\rm H} X(E) = \min\{d, \dim_{\rm H}^{\rho} E\}, \qquad \text{a.s.} \tag{10}$$

A uniform version of (10) is proved by Wu and Xiao (2009).
Its proof relies on SLND.

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3.3 Hausdorff dimension of $X^{-1}(F)$

Theorem 3.5 (Biermé, Lacaux and X., 2009)

For any Borel set $F \subseteq \mathbb{R}^d$ such that $\sum_{j=1}^{N} \frac{1}{H_j} > d - \dim_{\rm H} F$, $\dim_{\rm H} X^{-1}(F)$ equals

$$\min_{1\le k\le N} \left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\rm H} F) \right\}$$

$$= \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\rm H} F),$$

$$\text{if } \sum_{j=1}^{k-1} \frac{1}{H_j} \le d - \dim_{\rm H} F < \sum_{j=1}^{k} \frac{1}{H_j}.$$

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Corollary 3.6

If $\sum_{j=1}^{N} \frac{1}{H_j} > d$, then for every $x \in \mathbb{R}^d$,

$$\dim_{\mathrm{H}} X^{-1}(x) = \min_{1 \le k \le N} \left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d \right\}$$

$$= \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k d,$$

$$\text{if } \sum_{j=1}^{k-1} \frac{1}{H_j} \le d < \sum_{j=1}^{k} \frac{1}{H_j}.$$

Question: Find an exact Hausdorff measure function for $X^{-1}(x)$.

Borel measure on $X^{-1}(x)$: local times of X

Theorem 3.7 (Ayache, Wu and X. 2008; Wu and X., 2011)

If $\sum_{j=1}^{N} \frac{1}{H_j} > d$, then

- X has a local time $L(x,t)$ which is continuous for $(x,t) \in \mathbb{R}^d \times \mathbb{R}^N$.
- for every $x \in \mathbb{R}^d$, the measure $L(x,\cdot)$ is supported by $X^{-1}(x)$ and for all $t \in \mathbb{R}^N$ and $r > 0$ small enough,

$$L(x, B(t,r)) \le c_{3,5}\, r^{\beta} \big(\log\log(1/r)\big)^{\eta} := \psi(r),$$

where $\beta = \dim_{\rm H} X^{-1}(x)$ and η is a constant.

This implies $X^{-1}(x)$ has positive ψ-Hausdorff measure.

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4. Spectral condition for strong local nondeterminism

For any $\lambda \in \mathbb{R}^N$ and $h > 0$, denote by $C(\lambda, h)$ the cube with side-length $2h$ and center λ, i.e.,

$$C(\lambda, h) = \left\{x \in \mathbb{R}^N : |x_j - \lambda_j| \le h,\ j = 1, \cdots, N\right\}.$$

Let $L^2(C(0,T))$ be the subspace of $g \in L^2(\mathbb{R}^N)$ whose support is contained in $C(0,T)$.

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Theorem 4.1 (Luan and X., 2010)

Let $\{Y(t), t \in \mathbb{R}^N\}$ be a real, centered Gaussian field with stationary increments and $Y(0) = 0$. If for some $h > 0$ the spectral measure Δ of Y satisfies

$$0 < \liminf_{\|\lambda\|\to\infty} \rho(0, \lambda)^{Q+2} \Delta(C(\lambda, h)) \leq \limsup_{\|\lambda\|\to\infty} \rho(0, \lambda)^{Q+2} \Delta(C(\lambda, h)) < \infty, \qquad (11)$$

then for any $T > 0$ such that $ThN < \log 2$, for all $u, t^1, \ldots, t^n \in C(0, T)$,

$$\mathrm{Var}\Big(Y(u) \,\Big|\, Y(t^1), \ldots, Y(t^n)\Big) \geq c_{4,1} \min_{0\leq k\leq n} \rho(u,\, t^k)^2,$$

Examples

Example 4.1. Let $\{\xi_n, n \in \mathbb{Z}^N\}$ and $\{\eta_n, n \in \mathbb{Z}^N\}$ be two independent sequences of i.i.d. $N(0,1)$ random variables. Let

$$Z(t) = \sum_{n \in \mathbb{Z}^N} a_n \big(\xi_n \cos\langle n, t\rangle + \eta_n \sin\langle n, t\rangle\big), \quad t \in \mathbb{R}^N,$$

where $\{a_n, n \in \mathbb{Z}^N\}$ is a sequence of real numbers such that

$$a_n^2 \asymp \frac{1}{\big(\sum_{j=1}^N |n_j|^{H_j}\big)^{Q+2}}.$$

Then the Gaussian field $Y(t) = Z(t) - Z(0)$ has the same local properties as the Gaussian fields considered in the previous sections.

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Other examples include

- the solution of fractional stochastic heat equation on the circle (at fixed time t); see Tindel, Tudor and Viens (2003, 2004), Nualart and Viens (2009).
- spherical fractional Brownian motion, Istas (2005).

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Thank you