Lipschitz Equivalence of Fractals Generated by Nested Cubes (Math. Z., in press)

Ying Xiong (joint with Lifeng Xi)

xiongyng@gmail.com

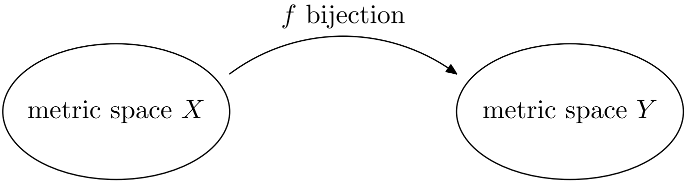
Department of Mathematics
South China University of Technology

Fractals and Related Fields II

June 12–17, 2011

Ile de Porquerolles, South of France

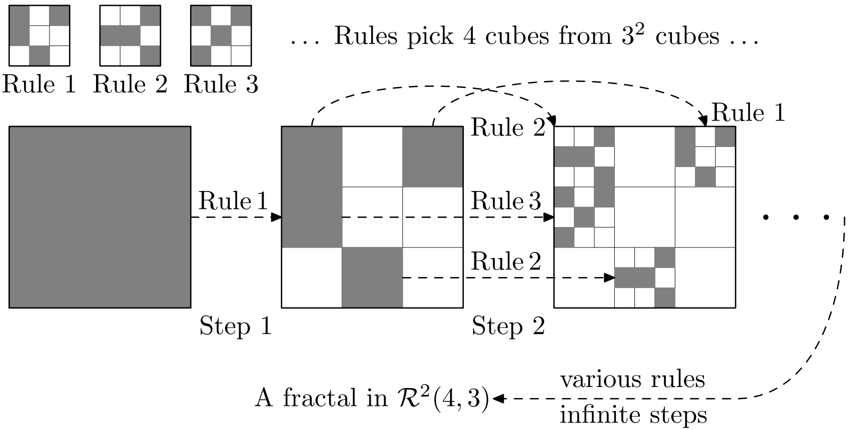
Lipschitz equivalence ($X \simeq Y$)



$$C^{-1} \leq \frac{d_Y\big(f(x_1), f(x_2)\big)}{d_X(x_1, x_2)} \leq C$$

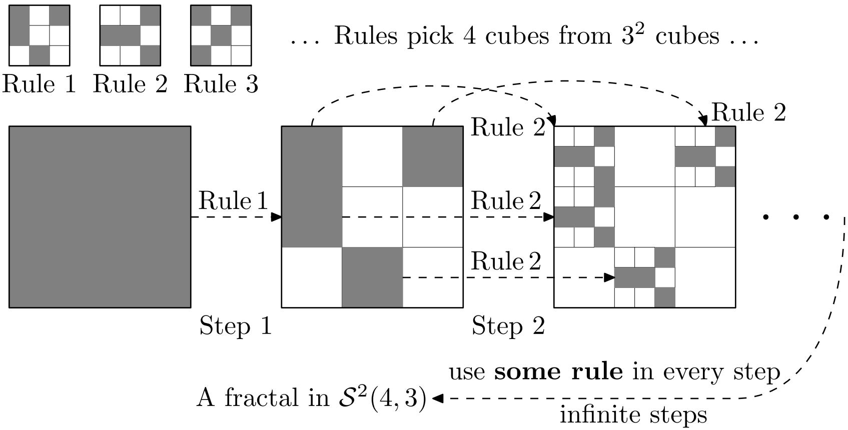
$X \simeq Y \Longrightarrow X, Y$ have same geometric structure

Rule, Nested Cubes & $\mathcal{R}^d(n, m)$



$\mathcal{R}^d(n, m)$: all such fractals whose rules pick n cubes from m^d cubes in $\mathbb{R}^d$.

A subclass $\mathcal{S}^d(n, m) \subset \mathcal{R}^d(n, m)$



$\mathcal{S}^d(n, m)$: all fractals in $\mathcal{R}^d(n, m)$ such that use same rule in every step.

More special, always use one rule $\longrightarrow$ self-similar set.

Lipschitz equivalence of fractals in $\mathcal{R}^d(n, m)$

Problem (Proposed by G. David and S. Semmes)

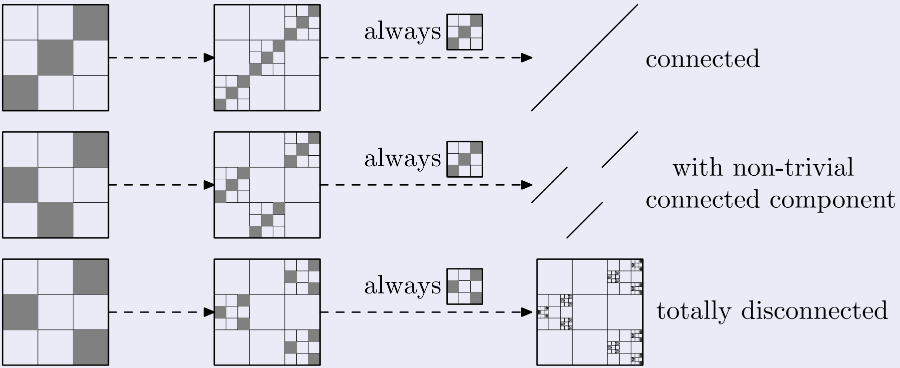
How to characterize Lipschitz equivalence for fractals in $\mathcal{R}^d(n, m)$?

Lipschitz equivalence of fractals in $\mathcal{R}^d(n,m)$

Problem (Proposed by G. David and S. Semmes)

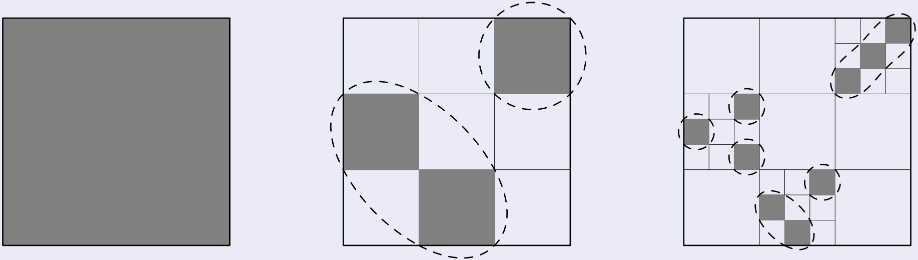
How to characterize Lipschitz equivalence for fractals in $\mathcal{R}^d(n,m)$?

Connected, disconnected or totally disconnected



Bounded block size (I)

Block of order k: connected component in k-th step structure.



Blocks of order 1

Blocks of order 2

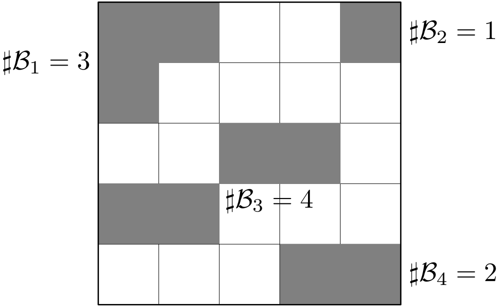
$$\{\text{Blocks of } E\} = \bigcup_{k\geq 1} \{\text{Blocks of order } k\}, \quad \text{for } E \in \mathcal{R}^d(n,m).$$

Bounded block size (II)

Definition (Bounded block size)

We say that $E \in \mathcal{R}^d(n, m)$ has bounded block size if $\exists M > 0$ such that

$$\sharp\mathcal{B} \leq M, \quad \text{for any block } \mathcal{B} \text{ of } E.$$

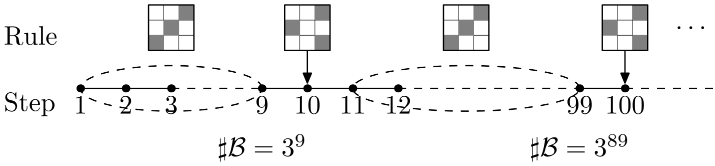


Bounded block size (III)

- Bounded block size is Lipschitz invariant.

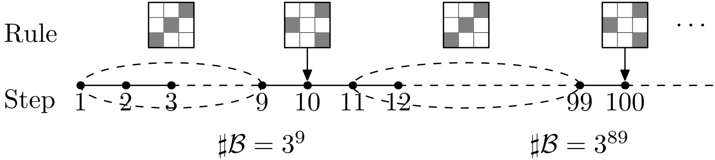
Bounded block size (III)

- Bounded block size is Lipschitz invariant.
- Bounded block size is stronger than totally disconnected.

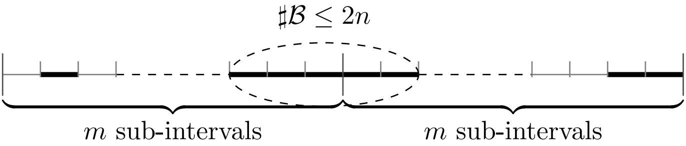


Bounded block size (III)

- Bounded block size is Lipschitz invariant.
- Bounded block size is stronger than totally disconnected.

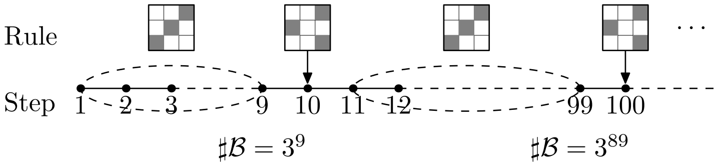


- All fractals in $\mathcal{R}^1(n,m)$ have bounded block size when $m > n$.

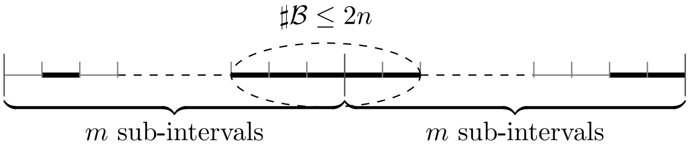


Bounded block size (III)

- Bounded block size is Lipschitz invariant.
- Bounded block size is stronger than totally disconnected.



- All fractals in $\mathcal{R}^1(n, m)$ have bounded block size when $m > n$.



- All totally disconnected self-similar sets in $\mathcal{R}^d(n, m)$ have bounded block size.

The result about $\mathcal{R}^1(n, m)$

Theorem

The cardinality of Lipschitz equivalence classes in $\mathcal{R}^1(n, m)$ is $2^{\aleph_0}$ when $m > n \geq 2$.

The result about $\mathcal{R}^1(n, m)$

Theorem

The cardinality of Lipschitz equivalence classes in $\mathcal{R}^1(n, m)$ is $2^{\aleph_0}$ when $m > n \geq 2$.

Remarks

- All fractals in $\mathcal{R}^1(n, m)$ have bounded block size when $m > n$.

The result about $\mathcal{R}^1(n, m)$

Theorem

The cardinality of Lipschitz equivalence classes in $\mathcal{R}^1(n, m)$ is $2^{\aleph_0}$ when $m > n \geq 2$.

Remarks

- All fractals in $\mathcal{R}^1(n, m)$ have bounded block size when $m > n$.
- The cardinality of the closed sets in Euclidean spaces is $2^{\aleph_0}$.

The result about $\mathcal{R}^1(n, m)$

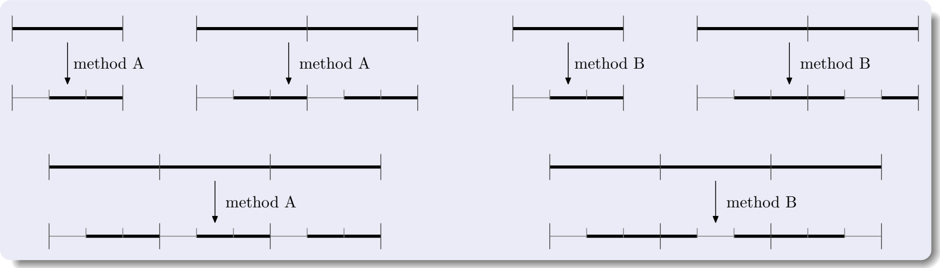
Theorem

The cardinality of Lipschitz equivalence classes in $\mathcal{R}^1(n, m)$ is $2^{\aleph_0}$ when $m > n \geq 2$.

Remarks

- All fractals in $\mathcal{R}^1(n, m)$ have bounded block size when $m > n$.
- The cardinality of the closed sets in Euclidean spaces is $2^{\aleph_0}$.
- The situation in higher dimension is more complicated.

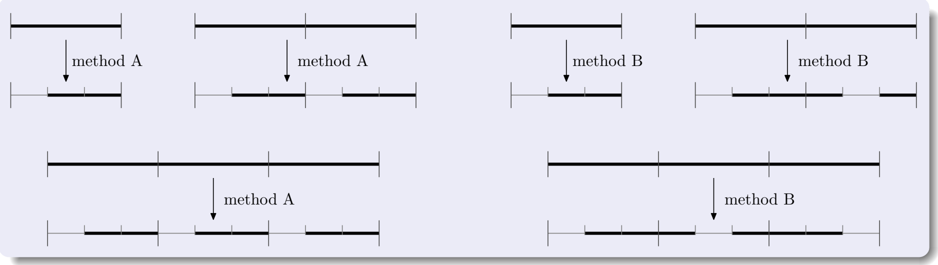
Idea of the proof for $\mathcal{R}^1(2,3)$



Method A $\xrightarrow{\text{several steps}}$ all touching two intervals

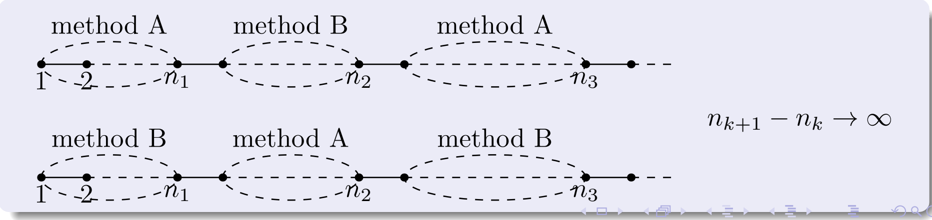
Method B $\xrightarrow{\text{several steps}}$ all touching three intervals only one exception

Idea of the proof for $\mathcal{R}^1(2,3)$



Method A $\xrightarrow{\text{several steps}}$ all touching two intervals

Method B $\xrightarrow{\text{several steps}}$ all touching three intervals only one exception



Fractals in $\mathcal{S}^d(n, m)$ of bounded block size

Let $\Omega(n, m) = \{1, \ldots, n\}^{\mathbb{N}}$ and $\rho(\omega, \omega') = m^{-\inf\{k:\, \omega_k \neq \omega'_k\}}$.

Theorem

- *Suppose that $E \in \mathcal{R}^d(n, m)$. Then*

$$E \simeq \Omega(n, m) \Longrightarrow E \text{ has bounded block size.}$$

- *Suppose that $E \in \mathcal{S}^d(n, m)$. Then*

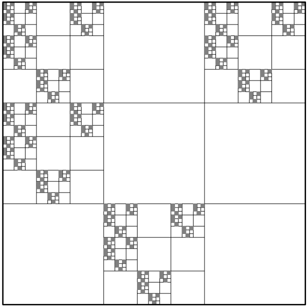
$$E \simeq \Omega(n, m) \Longleftrightarrow E \text{ has bounded block size.}$$

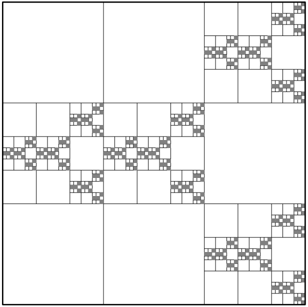
Totally disconnected self-similar sets in $\mathcal{S}^d(n, m)$

Corollary

E, F: totally disconnected self-similar sets in $\mathcal{S}^d(n, m)$. Then

$$E \simeq F \simeq \Omega(n, m).$$

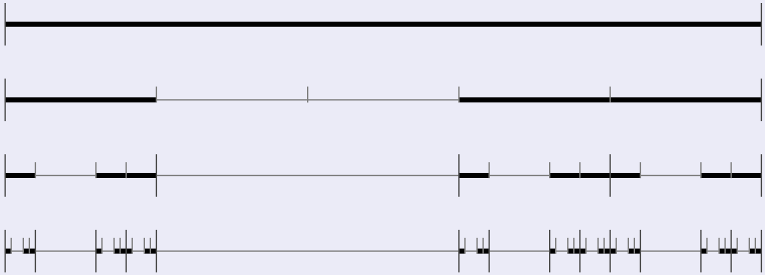




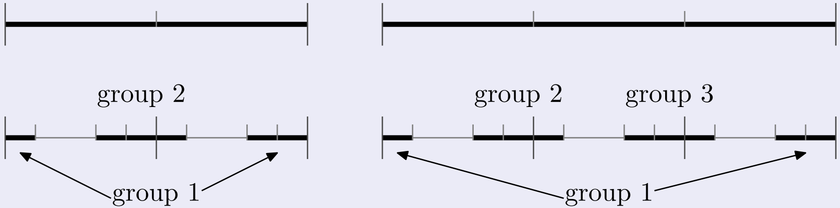
The two self-similar sets are Lipschitz equivalent.

Idea of the proof: equal grouping

$\{1,4,5\}$-set: $E_{1,4,5} = E_{1,4,5}/5 \cup (E_{1,4,5}+3)/5 \cup (E_{1,4,5}+4)/5$

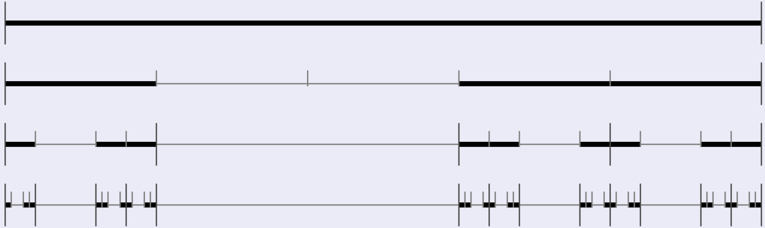


Equal grouping



Equal grouping is not Lipschitz invariant

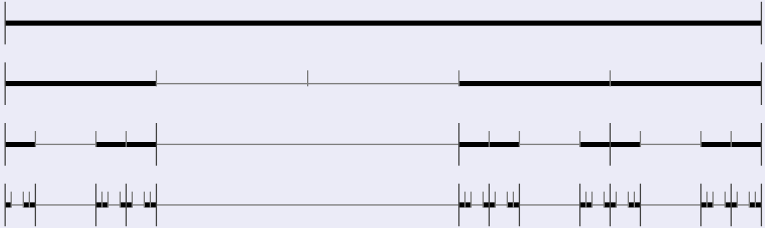
$\{1,-4,5\}$-set: $E_{1,-4,5} = E_{1,-4,5}/5 \cup (-E_{1,-4,5}+4)/5 \cup (E_{1,-4,5}+4)/5$



- Equal grouping does not hold for $E_{1,-4,5}$
- $E_{1,4,5} \simeq E_{1,-4,5} \simeq \Omega(3,5)$
- Equal grouping is not Lipschitz invariant

Equal grouping is not Lipschitz invariant

$\{1, -4, 5\}$-set: $E_{1,-4,5} = E_{1,-4,5}/5 \cup (-E_{1,-4,5} + 4)/5 \cup (E_{1,-4,5} + 4)/5$



- Equal grouping does not hold for $E_{1,-4,5}$
- $E_{1,4,5} \simeq E_{1,-4,5} \simeq \Omega(3, 5)$
- Equal grouping is not Lipschitz invariant

Problem (Proposed by Z. Y. Wen)

What fractals in $\mathcal{R}^d(n, m)$ are Lipschitz equivalent to $\Omega(n, m)$?

Bibliography

G. David and S. Semmes, Fractured fractals and broken dreams: self-similar geometry through metric and measure, Oxford Univ. Press, 1997.

K. J. Falconer and D. Marsh, On the Lipschitz equivalence of Cantor sets. Mathematika **39** (1992), no. 2, 223–233.

H. Rao, H.-J. Ruan and L.-F. Xi, Lipschitz Equivalence of Self-similar Sets, C. R. Acad. Sci. Paris Sér. I Math. I **342** (2006) 191–196.

L.-F. Xi and Y. Xiong, Self-similar sets with initial cubic patterns, C. R. Math. Acad. Sci. Paris, Sér. I Math. I **348** (2010) 15–20.

Y. Xiong and L.-F. Xi, Lipschitz equivalence of graph-directed fractals, Studia Math., **194** (2009), no.2, 197–205.

Z.-Y. Zhu, Y. Xiong and L.-F. Xi, Lipschitz equivalence of self-similar sets with triangular pattern, Sci. China Ser. A, **54** (2011), no.5, 1019–1026.