Local and global curvatures for classes of fractals

M. Zähle

Friedrich Schiller University Jena

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References

S. Winter, *Curvature measures and fractals*, Dissertationes Math. **453**, 2008.

(deterministic self-similar sets with polyconvex neighborhoods, curvature measures)

M. Zähle: *Lipschitz-Killing curvatures of self-similar random fractals*, TAMS **363**, 2011.

(self-similar random sets with singular neighborhoods, total curvatures)

S.Winter, M. Zähle: *Fractal curvature measures of self-similar sets*, Adv. in Geom. (to appear).

(deterministic measure version for singular neighborhoods)

J. Rataj, M. Zähle: *Curvature densities of self-similar sets*, Preprint.

(dynamical approach to local and global curvatures)

T. Rothe, M. Zähle: *Curvature-direction measures of self-similar sets*, Preprint.

(extension to non-isotropic quantities, new short proof for the measure versions)

Earlier and recent literature for the case of the Minkowski content: Lapidus, Falconer, Gatzouras, Rataj/Winter, Kesseböhmer/Kombrink, ...

1. Lipschitz-Killing curvature measures in classical (singular) curvature theory in $\mathbb{R}^d$

Notation

$$C_k(K, \cdot)\,, \quad k = 0, \dots, d$$

total curvatures: $C_k(K) = C_k(K, \mathbb{R}^d)$

Special cases $k = 0$: total Gauss curvature = Euler characteristic,
$k = d$: volume (for completeness)

Convex geometry *(Steiner, Minkowski, Hadwiger, Santalo, ..., Groemer, Schneider)*
$C_k(K)$ kth intrinsic volume of a (poly)convex body K

Differential and integral geometry *(Weyl, Chern, Blaschke, Santalo, ..., Wintgen, Cheeger/Müller/Schrader)*
$C_k(K, \cdot)$ in terms of integrating the traces of powers of the Riemannian curvature tensor over a C^2-manifold K and integrating the higher order mean curvatures over the boundary ∂K

Geometric measure theory - extension of both approaches *([Federer 1959], explicit representation [Z. 1986])*

integrals of kth generalized mean curvatures over the unit normal bundle $\mathrm{nor} K$ of a set K with positive reach (unique foot point property)

$$\widetilde{C}_k(K,\cdot) := \int_{\mathrm{nor} K \cap ((\cdot)\times S^{d-1})} S_{d-1-k}(\varkappa_1,\dots,\varkappa_{d-1})\, d\mathcal{H}^{d-1}$$

kth Lipschitz-Killing curvature measure on $\mathbb{R}^d$, $k=0,\dots,d-1$, where

$$S_l((\varkappa_1,\dots,\varkappa_{d-1}) := \mathrm{const}(d,l) \sum_{1\le i_1 \dots \le i_l \le d-1} \varkappa_1 \dots \varkappa_l$$

lth symmetric function of generalized principal curvatures $\varkappa_1,\dots,\varkappa_{d-1}$

For $\varepsilon > 0$ and $K \subset \mathbb{R}^d$ denote

$$K_\varepsilon := \{x \in \mathbb{R}^d : \operatorname{dist}(x, K) \le \varepsilon\}.$$

Theorem (Fu 1985)

For any compact $K \subset \mathbb{R}^d$ with $d \le 3$, Lebesgue-a.e. $\varepsilon > 0$ is a regular value of the distance function of K and, hence, the closure of the complement of the the parallel set K_ε has positive reach.

For arbitrary d and compact K with this property define the kth Lipschitz-Killing curvature measure of the parallel sets K_ε for such ε by

$$C_k(K_\varepsilon, \cdot) := (-1)^{d-1-k} C_k\left(\overline{(K_\varepsilon)^c}, \cdot\right)$$

(consistent definition).

For classical sets K as above we have

$$(w) \lim_{\varepsilon \to 0} C_k(K_\varepsilon, \cdot) = C_k(K, \cdot),$$

for fractal sets explosion. Therefore rescaling is necessary:

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2. Fractal curvatures - approximation by close neighborhoods

F self-similar (random) set in $\mathbb{R}^d$ with Hausdorff dimension D satisfying SOSC

Under the additional assumption on the regularity of the neighborhoods F_ε the following limits exist(almost surely):

$$C_k^{frac}(F) := \lim_{\varepsilon \to 0} \varepsilon^{D-k} C_k(F_\varepsilon)$$

in the "non-arithmetic case" and

$$C_k^{frac}(F) := \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{D-k} C_k(F_\varepsilon) \frac{1}{\varepsilon} d\varepsilon .$$

in general.

(Integral representation for $C_k(F)$ which admits some explicit or numerical calculations.)

New system of geometric parameters, allows to distinguish self-similar fractals with equal Hausdorff dimension, but different geometric features.

Measure version:

$$\begin{aligned} C_k^{frac}(F,\cdot) := &\ (w)\lim_{\delta\to 0}\frac{1}{|\ln\delta|}\int_\delta^1 \varepsilon^{D-k}C_k(F_\varepsilon,\cdot)\,\frac{1}{\varepsilon}\,d\varepsilon \\ = &\ C_k(F)\mathcal{H}^D(F)^{-1}\,\mathcal{H}^D(F\cap(\cdot))\,. \\ = &\ (w)\lim_{\varepsilon\to 0}\varepsilon^{D-k}C_k(F_\varepsilon,\cdot) \end{aligned}$$

in the non-arithmetic case.

Interpretation of the factors $\mathcal{H}^D(F)^{-1}C_k(F)$: some fractal analogues of the **pointwise mean curvatures** on smooth submanifolds, here: constant values because of self-similarity,

Main tool and additional result: interpretation of these factors as curvature densities, permits to consider other types of (random) fractals with scaling properties:

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3. Average curvature densities

Let O be from (SOSC), $SO := \bigcup_{i=1}^{N} S_i O$ for the generating similarities $S_1, \ldots, S_N$ with contraction rations $r_1, \ldots, r_N$.

For $a > 1$, $\varepsilon_0 > 0$ and $b := \max(2a, \varepsilon_0^{-1}|O|)$ let $\{A_F(x,\varepsilon) : x \in F,\, 0 < \varepsilon < \varepsilon_0,\ \}$ be a locally homogeneous neighborhood net:

$A_F(x,\varepsilon) \subset F_\varepsilon \cap B(x, a\varepsilon)$ and

$A_F(x,\varepsilon) = S_i(A_F(S_i^{-1}x, r_i^{-1}\varepsilon))$ if $x \in S_i F$ and $\varepsilon < b^{-1} d(x, \partial S_i(O))$ (homogeneity).

Examples:

1. $A_F(x,\varepsilon) = F_\varepsilon \cap B(x, a\varepsilon)$
2. $A_F(x,\varepsilon) = F_\varepsilon \cap \Pi_F^{-1}(B(x,\varepsilon))$,
   the set of those points from F_ε which have a foot point on F within the ball $B(x,\varepsilon)$
3. $A_F(x,\varepsilon) = \{y \in F_\varepsilon : |y - x| < \varrho_F(y,\varepsilon)\}$,
   where $\varrho_F(y,\varepsilon)$ is determined by $\mathcal{H}^D(F \cap B(y, \varrho_F(y,\varepsilon))) = \varepsilon^D$

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First main result for deterministic neighborhood-regular self-similar sets F:

For $\mathcal{H}^D$-a.a. $x \in F$ the following limit exists

$$\lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{-k} C_k\big(F_\varepsilon, A_F(x,\varepsilon)\big) \, \frac{1}{\varepsilon} \, d\varepsilon$$

and equals the constant

$$\mathcal{H}^D(F)^{-1} \Big(\sum_{i=1}^N r_i^D |\ln r_i| \Big)^{-1} \int_F \int_{\frac{d(y,\partial(SO))}{2a}}^{\frac{d(y,\partial O)}{2a}} \varepsilon^{-k} C_k\big(F_\varepsilon, A_F(y,\varepsilon)\big) \, \frac{1}{\varepsilon} \, d\varepsilon \, \mathcal{H}^D(dy)$$

provided the last double integral converges.

Analogous result for self-similar random sets F can be proved

Tools: properties of the Lipschitz-Killing curvature measures of the parallel sets and scaling of the self-similar (random) sets, associated dynamical systems, Birkhoff's ergodic theorem

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Main properties of the Lipschitz-Killing curvature measures:

1. $C_k(gK, g(\cdot)) = C_k(K, \cdot)$ for any Euclidean motion g (motion invariance)
2. $C_k(\lambda K, \lambda(\cdot)) = \lambda^k C_k(K, \cdot)\,,\ \lambda > 0$ (scaling of order k)
   $C_k(K, B) = C_k(K', B)$ if $K \cap G = K' \cap G$ and $B \subset G$ for some open set G (locally determined)

Self-similarity of the fractal set F and corresponding Hausdorff measure:

$$F = S_1(F) \cup \ldots \cup S_N(F)$$

for similarities $S_1, \ldots, S_N$ with contraction ratios $r_1, \ldots, r_N$ and (SOSC), Hausdorff dimension D: $\sum_{i=1}^{N} r^D = 1$,

self-similar measure: $\mu := (\mathcal{H}^D(F))^{-1}\mathcal{H}^D(F \cap (\cdot))$ satisfies

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Associated dynamical system:

"inverse" mapping of the IFS $S_1, \ldots, S_N$:

$$\Phi(x) := S_i^{-1}(x) \text{ if } x \in S_i(F)$$

is a.e. determined and the self-similar measure μ is invariant under Φ, moreover:

$$[F, \mu, \Phi] \text{ is an ergodic dynamical system}$$

Idea for proof of the theorem: translation of the curvature densities into the language of Birkhoff's ergodic theorem w.r.t. to this system using the above properties.

Consequences: Second main result

1. Derivation of average convergence of the rescaled global curvatures (by means of the above local convergence for the neighborhood net $\{A(x, \varepsilon)\}$ from Example 3 and Fubini-type arguments)
2. Derivation of average convergence of the rescaled curvature measures (short proof for curvature-direction measures: T. Rothe)

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From average to ordinary convergence

Former approaches: by means of the renewal theorem for the arithmetic and non-arithmetic case

Probable substitute in general: Birkhoff's ergodic theorem together with asymptotic properties of probability distributions in the non-arithmetic case,

Works in our case (for the distribution of the logarithmic contraction ratios of the similarities $S_1, \dots S_N$): in the non-arithmetic case ordinary convergence can be derived

Other classes: **self-similar random sets**, up to now only global curvatures, can be localized

V-variable random fractals: similar results as above (w.p.1) for average curvature densities, but with different rescaling,

(average) convergence of global curvatures, with the same rescaling, (with J. Hutchinson)

(stationary fractal tessellations with scaling properties, ...)

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