Fine inducing and rigidity of equilibrium measures for rational maps

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Plan

1. Basic facts about holomorphic endomorphisms of $\mathbb{P}^1$, topological entropy
2. Equilibrium measures
3. Results
4. Construction

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Endomorphisms

Space

$$\mathbb{P}^1 = \mathbb{C}^2 \setminus \{0\} / \sim$$

$$z \sim w \iff z = \lambda w, \lambda \in \mathbb{C}$$

Endomorphisms

$$F(z_0, z_1) = (f_0(z_0, z_1) f_1(z_0, z_1))$$

where f_i are homegenous polynomials of degree d, having no (nontrivial) common roots

$$f(z) = \frac{p(z)}{q(z)} \quad (1)$$

and $d = \max(\text{degp}, \text{degq})$.

Entropy

$$h_{\text{top}}(f) = \log(deg_{\text{top}}(f)) = k \log d$$

- "$\geq$" follows from a general result of [Misiurewicz, Przytycki]
- "$\leq$" proved by Gromov (uses specific structure of $\mathbb{P}^k$).

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Maximal entropy measure

Ruelle Operator

$$\mathcal{L} : C(J(f)) \to C(J(f))$$

$$\mathcal{L}(g)(x) = \sum_{y \in f^{-1}(x)} g(y)$$

Maximal measure

For every Borel measure ν on $J(f)$ the sequence

$$\frac{1}{d^n}(\mathcal{L}^*)^n(\nu) \to \mu$$

and μ is the unique measure of maximal entropy.

RPF Operator

$\phi : J(f) \to \mathbb{R}$ Hölder continuous. Assume that the topological pressure $P(\phi)$ satisfies

$$P(\phi) > \sup \phi$$

$$\mathcal{L}_\phi : C(J(f)) \to C(J(f))$$

$$\mathcal{L}_\phi(g)(x) = \sum_{y \in f^{-1}(x)} \exp \phi(y) g(y)$$

Equilibrium measure

Theorem: There exists a unique probability measure ('conformal measure') which is an eigenmeasure of the conjugate RFP operator (i.e $(\mathcal{L})^*(\nu) = \lambda\nu$) and a (unique) continuous, nonnegative function ρ -an eigenfunction of the normalized RPF operator $\frac{1}{\lambda}\mathcal{L}$. The measure

$$\mu_\phi = \rho\nu$$

is the unique equilibrium measure for ϕ.

Critically finite maps and the dimension of maximal entropy measure- known results

A map is called critically finite if the trajectories of all critical points are finite. To every critically finite map one can associate (in a canonical way) an orbifold. Critically finite maps with a parabolic orbifold (CFPO) are easily classified. Among polynomials, there are only two types of CFPO: $z \mapsto z^d$ and Chebyshev polynomials.

Theorem ([Z])

The dimension of the maximal entropy measure and the dimension of the Julia set coincide iff f is CFPO Moreover, if f is not CFPO then the maximal measure is singular with respect to the Hausdorff measure at its dimension.

Joint work with M. Szostakiewicz and M. Urbański

Fine inducing

The map f is replaced by an infinite Iterated Function System:

$$F : \bigcup U_i \to U$$

where $U_i \subset U$, $F_{|U_i} = f^{n(i)}$ is a conformal isomorphism onto U and $\mu_\phi(\bigcup U_i) = \mu_\phi(U)$.

Main estimate

Let V_N be the union of U_i's for which $n(i) > N$. Then

$$\mu_\phi(V_N) < C\theta^N$$

(where $0 < \theta < 1$ and $C > 0$ are some constants).

Dimension

Theorem 1: Let f be a rational map ϕ-an admissible potential, μ_ϕ- the unique equilibrium measure. If $\dim \mu_\phi = \dim J(f) = h$ then one of the following holds:

1. The closure of the postcritical set $P(f)$ is disjoint from $J(f)$ and ϕ is cohomologous to $-h \log |f'|$
2. $$\mathrm{card P(f) \cap J(f)} \leq 4.$$
   There are no other points in $\mathrm{cl} P(f) \cap J(f)$.

This generalizes the known result on the dimension of the maximal measure, and simplifies its proof.

Stochastic properties of μ_ϕ

Theorem 2: (Exponential decay of correlations for Hölder continuous 'observables')
For every $\alpha \leq 1$, every α-continuous function $\varphi : J(f) \to \mathbb{R}$, and every bounded measurable function $\psi : J(f) \to \mathbb{R}$

$$\left| \int \psi \circ f^n \cdot \varphi d\mu_\phi - \int \varphi d\mu_\phi \int \psi d\mu_\phi \right| = O(\theta^n)$$

with some constant $0 < \theta < 1$, depending on α.
This implies the CLT for Hölder continuous 'observables'.
This gives a straightforward proof of the result proved earlier by N. Haydn.

Real analyticity of the topological pressure

Theorem 3:
The topological pressure function

$$t \mapsto P(t\phi)$$

is real- analytic in some interval (t_1, t_2) (containing the value $t = 1$).

Good components

Lemma:
If U is an open topological disc containing no critical values of f^q then there exist families W_n of connected components of $f^{-qn}(U)$ such that $W_0 = \{U\}$ and

1. a(n) if $V \in W_{n+1}$ then $f^q(V) \in W_n$.
2. b(n) $f^{qn}_{|V}$ is univalent.
3. c(n) $\max(\operatorname{diam} V : V \in W_n) < C_q \gamma^n$
4. d(n) Let Z_n be the family of all connected components of $f^{-q}(V)$, $V \in W_{n-1}$. Let

$$B_n(x) = \sum_{y \in f^{-qn} x \cap V, V \in Z_n \setminus W_n} \exp S_n \tilde{\phi}_q(y)$$

Then

$$B_n(x) \leq \exp(-nq\theta)$$

where $\theta = \frac{\tilde{\theta}}{4}$.

Pullbacks

A pullback of length n is a sequence V_k of components of $h^{-k}(U)$, such that $h(V_{k+1}) = V_k$.

A pullback is *good* if V_n is a good component.

A pullback is *very good* if, additionally, the diameter of V_k is (much) smaller than the distance $\mathrm{dist} V_k, \partial U)$.

In particular, very good pullbacks are either totally contained in U or disjoint from U.

Induced map

Let $x \in U$. Define

$$F(x) = f^{n(x)}(x)$$

where $n(x)$ is the smallest positive integer n such that there exists *very good pullback* of length n mapping $h^n(x)$ to x.