Random matrices and their applications
October 8-10, 2012

Teo Banica

Block-modified Wishart matrices and free Poisson laws

We study the random matrices of type $\tilde{W} = (id \otimes \varphi)W$, where W is a complex Wishart matrix of parameters (dn, dm), and $\varphi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a self-adjoint linear map. We prove that, under suitable assumptions, we have the $d \to \infty$ eigenvalue distribution formula $\delta m \tilde{W} \sim \pi_{mn\rho} \boxtimes \nu$, where ρ is the law of φ, viewed as a square matrix, π is the free Poisson law, ν is the law of $D = \varphi(1)$, and $\delta = tr(D)$. (Joint work with Ion Nechita.)

Benoit Collins

Applications of Random matrices to quantum information theory

Random techniques are quite useful in classical information theory. In the same vein, it has been discovered in the last few years that random matrix theory can play a crucial role in the understanding of the behaviour of random quantum channels. I will review a few very important problems in quantum information theory, and outline the random matrix techniques that have lead to a solution to some of these problems. In turn, quantum matrix becomes a source of interesting new random matrix models with unexpected properties. This talk is partly based on joint work with the following coauthors: Ion Nechita, Serban Belinschi, Motohisa Fukuda, and Camille Male.

Romain Couillet

Robust estimation in the random matrix regime

Many subspace methods for signal processing rely on the sample covariance matrix (SCM) from n independent observations of some N-population vector. The SCM is however an inappropriate estimate of the population covariance when n is not large compared to N, which motivated the recent works on G-estimation. Similarly, the SCM may be inappropriate for non-Gaussian populations, for which robust estimators were put in place in the 70's. In this talk, we introduce first results on "robust G-estimation", where it is shown that, for a given population vector model, G-estimation can be applied on robust covariance estimates in place of the SCM.

Alan Edelman

Progress on the Random Matrix Method of Ghosts and Shadows for Beta Ensembles

In this talk, we will recall how the threefold way of real, complex, quaternion is not very special anymore, and how one can work with a continuum of beta. We will show how this method has yielded an algorithm that is matching the statistics for singular values of an m by n ghost Gaussian by a real diagonal. This is the Wishart ensemble with general covariance.

Maxime Février

Outliers in the spectrum of spiked deformations of unitarily invariant random matrices

We investigate the asymptotic behavior of the eigenvalues of the random matrix A+U*BU, where A and B are deterministic Hermitian matrices and U is drawn from the unitary group according to Haar measure. We discuss the existence and localization of "outliers", i.e. eigenvalues lying outside from the bulk of the spectrum. This is joint work with S. Belinschi, H. Bercovici and M. Capitaine.

Olivier Guédon

Concentration phenomena in high dimensional geometry

I will describe some results about concentration of volume of high dimensional convex bodies obtained in the last decade. Central limit theorem for convex bodies is one of the main achievement of these series of work. I will also present some open problems, like the thin shell conjecture, the problem of spectral gap, and the problem of approximation of the covariance matrix. Extension of these results to new classes of probability measures, like Cauchy measure or more generally κ-concave measures will be discussed.

Friedrich Götze

Local Universality of Repulsive Particle Systems and Random Matrices

We show that local correlation universality holds for deformations of the GUE ensemble which don't allow a spectral interpretation, but have a local repulsion exponent equal to that of GUE eigenvalues. This is joint work with M. Venker. Extensions of these results for β-ensembles will be discussed. We shall review as well recent joint results with A. Naumov and A. Tikhomirov for limit laws for general Wigner type ensembles with non identical distributed entries and martingale structure.

Uffe Haagerup

Random Matrices in Operator Algebra Theory - a Survey

Since Wigner's pioneering work from 1995 random matrices has been an important tool in Mathematical Physics. After Voiculescu in 1991-95 used random matrices to solve some deep open problems about von Neumann algebras, random matrices has also played a key role in operator algebra theory. In 2005 Steen Thorbjørnsen and the speaker were able to solve an old problem on C*-algebras, by making careful estimates of the largest and smallest eigenvalues in random ensamples, which can be expressed as (non-commutative) polynomials in two or more indepent GUE-random matrices. Shortly after similar estimates for polynomials in GOE- and GSE- matrices were obtained by Hanne Schultz (2006), but the corresponding problem for polynomials in two or more non-Gaussian selfadjoint random matrices with independent entries was solve only recently by Greg Anderson (2011). Related results have been obtained by Guionnet & Shlyakhtenko (2008), Male (2010) and Collins & Male (2011).

Michael Jordan

Matrix Concentration Inequalities via the Method of Exchangeable Pairs

I will present an approach to deriving exponential concentration inequalities and polynomial moment inequalities for the spectral norm of a random matrix based on Stein's method of exchangeable pairs. Our work extends results by Chatterjee on concentration for scalar random variables to the setting of random matrices. When applied to a sum of independent random matrices, this approach yields matrix generalizations of the classical inequalities due to Hoeffding, Bernstein, Khintchine, and Rosenthal. The same technique delivers bounds for sums of dependent random matrices and more general matrix-valued functions of dependent random variables. [Joint work with Lester Mackey, Richard Chen, Brendan Farrell and Joel Tropp].

Oleksiy Khorunzhiy

On spectral properties of large symmetric dilute random matrices

We consider an ensemble of $n \times n$ real symmetric matrices $H^{(n,\rho)}$ of the form

$$H_{ij}^{(n,\rho)} = a_{ij}\, b_{ij}^{(n,\rho)}, \quad 1 \le i \le j \le n, \tag{1}$$

where $\{a_{ij}, 1 \leq i \leq j\}$ and $\{b_{ij}^{(n,\rho)}, 1 \leq i \leq j \leq n\}$ are families of jointly independent random variables such that a_{ij} are of the same probability law that is symmetric and has several first moments finite, $V_k = \mathbf{E}|a_{ij}|^k < \infty$; random variables $b_{ij}^{(n,\rho)}$ are proportional to the Bernoulli ones, $b_{ij}^{(n,\rho)} = 1/\sqrt{\rho}$ with probability ρ/n and $b_{ij}^{(n,\rho)} = 0$ with probability $1 - \rho/n$, $0 < \rho \leq n$. We are interested in the asymptotic behavior of the spectral norm $\|H^{(n,\rho)}\| = \lambda_{\max}^{(n,\rho)}$ in the limit of infinite n and ρ.

Theorem 1. *Let the probability distribution of a_{ij} be such that for some $\phi > 0$ the moment $V_{12+2\phi}$ exists. Then for any sequence $\rho_n = n^{2/3(1+\varepsilon)}$ with given $\varepsilon > 3/(6+\phi)$, the limiting probability*

$$\limsup_{n\to\infty} \mathbf{P}\left(\lambda_{\max}^{(n,\rho_n)} \geq 2\sqrt{V_2}\left(1 + \frac{x}{n^{2/3}}\right)\right) \leq G(x), \quad x > 0 \tag{2}$$

admits a universal upper bound in the sense that $G(x)$ does not depend on the values of V_{2l}, $2 \leq l \leq 6$ and of $V_{12+2\phi}$; in particular, relation (2) is true with $G(x) = e^{-Cx^{3/2}}$.

The proof is based on the study of high moments $L_n = \mathbf{E}\sum_{i=1}^{n}(\hat{H}_n^{2s_n})_{ii}$, $s_n = \chi n^{2/3}$, $\chi > 0$ of matrices $\hat{H}_n = \hat{H}^{(n,\rho_n)}$ of the form (1) with truncated random variables $\hat{a}_{ij}$. We study L_n with the help of a generalization [1] of the approach developed in [2, 3].

The further studies show that if $\rho_n = O(n^{2/3})$, then the averages L_n change their asymptotic behavior and the higher moments V_{2l} are involved into the corresponding upper bounds. This means that the value $n^{2/3}$ can be the critical one for the upper bounds of the form (2). We also discuss the sub-critical asymptotic regime $\rho_n = o(n^{2/3})$ and describe generalizations of the Catalan numbers related with the asymptotic expansions of L_n.

References

[1] Khorunzhiy, O. *Random Oper. Stoch. Equ.*, **20**, 2012, 25-68

[2] Sinai, Ya., Soshnikov, A. *Bol. Soc. Brasil. Mat.*, **29**, 1998, 1-24

[3] Soshnikov, A. *Commun. Math. Phys.*, **207**, (1999), 697-733

Arno Kuijlaars

Multiple orthogonal polynomials and the normal matrix model

The normal matrix model is a random matrix model defined on complex matrices. The eigenvalues in this model fill a two-dimensional domain in the complex plane as the size of the matrices tends to infinity. Orthogonal polynomials with respect to a planar measure are a main tool in the analysis. In many interesting cases, however, the orthogonality is not well-defined, since the integrals that define the orthogonality are divergent. I will discuss a way to redefine the orthogonality which leads to multiple orthogonal polynomials. For the special case of a monomial potential it is possible to do a complete steepest descent analysis of the associated Riemann-Hilbert problem which leads to strong asymptotics of the multiple orthogonal polynomials, and in particular to the twodimensional domain on which the eigenvalues accumulate. This is joint work with Pavel Bleher (Indianapolis).

Philippe Loubaton

Applications of Gaussian Information plus Noise models to source localization

The purpose of this talk is to show how the asymptotic behaviour of large information plus noise models can be used in order to study the so-called source localization problem when the number of sensors and the number of observations are of the same order of magnitude. In this context, the observation is a $M \times N$ random matrix $\mathbf{Y}_N$ given by

$$\mathbf{Y}_N = \mathbf{B}_N + \mathbf{V}_N$$

where $\mathbf{B}_N$ is a deterministic $M \times N$ matrix representing the contribution of useful signals and where $\mathbf{V}_N$ is a random matrix with zero mean and variance σ^2 i.i.d. complex Gaussian entries modelling an additive noise. It is assumed that

- $\text{Rank}(\mathbf{B}_N) = K < M$
- The column space of $\mathbf{B}_N$ coincides with the space generated by K vectors $(\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K))$ where function $\theta \to \mathbf{a}(\theta)$ is given by

  $$\mathbf{a}(\theta) = \frac{1}{\sqrt{M}}(1, e^{i\theta}, \ldots, e^{i(M-1)\theta})^T$$

  where the parameters $(\theta_k)_{k=1,\ldots,K}$ are unknown.

In the context of source localization, K represent the number of sources, the parameters $(\theta_k)_{k=1,\ldots,K}$ characterize the direction of arrivals of the sources, while M and N coincide with the number of sensors and the number of observations respectively.

The source localization problem consists in estimating the parameters $(\theta_k)_{k=1,\ldots,K}$ from the observation $\mathbf{Y}_N$. The most popular estimators, called subspace estimators, are based on the property that if $\mathbf{\Pi}_N$ represents the orthogonal projection matrix on the column space of $\mathbf{B}_N$, then, the directional parameters $(\theta_k)_{k=1,\ldots,K}$ coincide with the arguments of the maximum of $\theta \to \eta_N(\theta)$ where $\eta_N(\theta)$ is defined by

$$\eta_N(\theta) = \mathbf{a}(\theta)^* \mathbf{\Pi}_N \mathbf{a}(\theta) \tag{1}$$

In practice, the parameters $(\theta_k)_{k=1,\ldots,K}$ are estimated as the arguments of the K greatest local maxima of a function $\theta \to \hat{\eta}_N(\theta)$ where $\hat{\eta}_N(\theta)$ represents for each θ a well chosen estimate of $\eta_N(\theta)$.

The derivation of consistent subspace estimators is quite well understood when $N \to +\infty$ and M remains fixed, a regime which appears to be relevant if the number of observations N is much larger than the number of sensors M. In order to address practical scenarii where M and N are of the same order to magnitude, the behaviour of various estimation schemes will be studied in the case where M and N converge to $+\infty$ in such a way that $\frac{M}{N} \to c$ where $0 < c < +\infty$. 3 cases will be considered:

- K and the parameters $(\theta_k)_{k=1,\ldots,K}$ do not scale with (M, N). In this context, any reasonable estimator is consistent, and for each k, $N^{3/2}(\hat{\theta}_k - \theta_k)$ has a Gaussian behaviour.
- K does not scale with (M, N), but some of the parameters, say θ_1 and θ_2, verify $\theta_1 - \theta_2 = O(\frac{1}{N})$. This context is of practical interest because it is a model for situations in which certain directional parameters are close. It will be shown that in contrast with the above case, the usual estimators derived in the regime M fixed fail to estimate θ_1 and θ_2, but that it is still possible to derive consistent estimators such that $N^{3/2}(\hat{\theta}_k - \theta_k)$ has a Gaussian behaviour for $k = 1, 2$.
- The case where K scales with (M, N) will also be considered.

These results are essentially based on the analysis of the behaviour of the eigenvalues and the eigenvectors of the empirical matrix $\frac{\mathbf{Y}_N \mathbf{Y}_N^*}{N}$ when M and N converge to $+\infty$ in such a way that $\frac{M}{N} \to c$. Various numerical experiments will illustrate the above results.

Camile Male

The strong asymptotic freeness of large random matrices

Motivated by Operator Algebraic questions, Voiculescu introduced Free Probability in the early eighties. Ten years later, he discovered that Free Probability explains some behaviours of eigenvalues of Random Matrices in the limit of large dimension. In this talk I will describe recent convergence results for norms for random matrices with unitary invariance, hereby solving conjectures by Pisier and Haagerup. This enlarges the class of matrices for which Bai-Silverstein "no eigenvalues outside a neighborhood of the limiting spectrum" is known. In collaboration with Benoît Collins.

Kenneth Maples

Universality of some p-adic matrix models

Recent work in number theory has explored how certain statistical phenomena on factorization in number fields could be explained in terms of random matrices over non-archimedean fields. Likewise, invariants of random graphs can be derived from their adjacency matrices modulo different primes. In this talk I will discuss two methods which can establish universality in these and related situations.

Ralf Müller

Analysis of a combinatorial optimization problem in wireless communications

We consider the minimum of the quadratic form x'Jx for x in D^K for a non-convex set D as K grows to infinity where J is a unitarily invariant random matrix with asymptotic eigenvalue distribution specified by its R-transform R(w). Using replica calculus from statistical physics, we find a general from to express the minimum by means of the free Fourier transform (Harish-Chandra-Itzykson-Zuber integral). In the given application (vector precoding in wireless communication), the minimum turns out to be well approximated by the 1-step replica symmetry breaking ansatz, while the replica symmetric ansatz fails for random matrices whose conditioning number is large. Furthermore, we characterize statistical properties of the solution vector x in the limit of large K.

Maciej Nowak

Spectral shock waves in dynamical random matrix theories

We obtain several classes of non-linear partial differential equations for various random matrix ensembles undergoing Brownian type of random walk. These equations for spectral flow of eigenvalues as a function of dynamical parameter ("time") are exact for any finite size N of the random matrix ensemble and resemble viscid Burgers-like equations known in the theory of turbulence. In the limit of infinite size of the matrix, these equations reduce to complex inviscid Burgers equations, proposed originally by Voiculescu in the context of free processes. We identify spectral shock waves for these equations in the limit of the infinite size of the matrix, and then we solve exact, finite N nonlinear equations in the vicinity of the shocks, obtaining in this way universal, microscopic scalings equivalent to Airy, Bessel and cuspoid kernels.

Leonid Pastur

Eigenvalues of Certain Random Matrices Arising in Quantum Informatics

We consider two ensembles of random matrices related to certain problems of quantum informatics. The first ensemble is a generalization of the Wishart Ensemble viewed as the sum of independent rank-one operators. However, in contrast to the Wishart Ensemble the corresponding independent random vectors are the tensor products of a fixed number p ≥ 2 of other independent vectors. The second ensemble is also similar to the Wishart Ensemble viewed as the product of matrix with independent entries and the transposed (hermitian conjugate) matrix. However, in contrast to the Wishart Ensemble the corresponding matrix is triangular. We show that the limiting Normalized Counting Measures can be found from certain functional equations and discuss their new properties.

Mariya Shcherbina

β matrix models in the multi cut regime

We consider β-matrix models with real analytic potentials for multi-cut regime. The recent results on the asymptotic behavior of the characteristic functional of linear eigenvalue statistics, in particular, non gaussian behavior of the characteristic functional in the multi-cut regime will be discussed. We discuss also the construction of the asymptotic expansion in n^{-1} and the applications of these results to the proof of the universality conjecture for β-matrix models.

Alexander Soshnikov

Central limit theorem type results in large random matrices with independent entries

In the first part of the talk I will discuss CLT type results for linear statistics in Wigner and band random matrices. The second part of my talk will be devoted to results about the distribution of the outliers in the spectrum.

Alexander Tikhomirov

Limit theorems for spectrum of products of large random matrices

We explain some results obtained jointly with F. Götze in the last years. We consider the limit theorems for the spectrum of product of large random matrices from different ensembles. First we consider the generalization of Marchenko–Pastur law for the distribution of singular values of product of rectangular matrices. We consider the generalization of circular law as well. It will be shown that under condition that random variables have uniformly integrated second moment the limit distribution of product of independent matrices from Girko–Ginibre Ensemble is the power of uniform distribution on the unit disc of complex plane. We show as well that the same result still true for product of Wigner matrices and for matrices with pairwise correlated entries. We describe some general approach to investigation of asymptotic behavior of singular values and eigenvalues of product of large random matrices.

Jianfeng Yao

Recent statistical applications from large sample covariance matrices

Since the research interests of the audience of this workshop seem highly concentrated on the mathematical theory of random matrices, this expository talk is aimed at explaining how this beautiful theory can help to solve some statistical problems where a large number of variates is involved. Two of such problems will be discussed. The first concerns hypothesis testing about population covariance matrices. Here, the main tools are central limit theorems for eigenvalue statistics of large sample covariance matrices or large Fisher matrices. The second concerns the estimation of the population spectral distribution from the (observed) sample covariance matrix. Some application to large-dimensional time series will be given. The talk is based on the papers listed below. References:

- Z. D. Bai, D. Jiang, J. Yao and S. Zheng, 2009. Corrections to LRT on Large Dimensional Covariance Matrix by RMT. Ann. Statistics, 37 (6B) , 3822–3840
- W.M. Li and J.F. Yao, 2011. A local moments estimation of the spectrum of a large dimensional covariance matrix. Submitted.
- W.M. Li and J.F. Yao, 2012. A contour-integral based estimation of a population spectral distribution from high-dimensional data Submitted.